TOPOLOGIE IV – EXERCISE SHEET 6

MARCUS NICOLAS

The obstruction to orientability. Applying $\mathrm{B}(-)$ to the cartesian square of ∞-groups

$$\begin{array}{ccc} \mathrm{SG} & \longrightarrow & \mathrm{G} \\ \downarrow & \lrcorner & \downarrow \\ * & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \end{array}$$

yields a cartesian square of pointed groupoids

$$\begin{array}{ccc} \mathrm{BSG} & \longrightarrow & \mathrm{BG} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \theta} \\ * & \longrightarrow & \mathrm{K}(\mathbb{Z}/2\mathbb{Z}, 1) \end{array}$$

By definition, the pullback $\xi^*\theta$ of θ along a stable spherical fibration $\xi\colon B \to \mathrm{BG}$ is exactly the obstruction to the existence of a lift

$$\begin{array}{ccc} & & \mathrm{BSG} \\ & \nearrow & \downarrow \\ B & \xrightarrow[\xi]{} & \mathrm{BG} \end{array}$$

or in other words, the obstruction to orientability. Since there exists non-orientable spherical fibrations, the class $\theta \in \mathrm{H}^1(\mathrm{BG}; \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$ cannot vanish. Therefore $\theta = w_1$.

Stable Hurewicz. If X is a spectrum, there is a natural comparison morphism

$$\pi_0(X \otimes \mathbb{S}) \to \pi_0(X \otimes \mathbb{Z})$$

induced by the ring map $\mathbb{S} \to \mathbb{Z}$.

- If $E \equiv \mathbb{S}$, then this map is the identification
$$\pi_0^{\mathrm{s}}\big(S^0\big) \simeq \mathrm{H}_0\big(S^0; \mathbb{Z}\big)$$
obtained by taking the colimit of the Hurewicz isomorphisms
$$\pi_n\big(S^n\big) \simeq \mathrm{H}_n\big(S^n; \mathbb{Z}\big) \simeq \mathrm{H}_0\big(S^0; \mathbb{Z}\big)$$
- Since $\pi_0\colon \mathrm{Gpd} \to \mathrm{Set}$ preserves finite products, the adjunction
$$\mathrm{Gpd} \underset{\longleftarrow}{\overset{\pi_0}{\underset{\perp}{\longrightarrow}}} \mathrm{Set}$$
induces
$$\mathrm{CGrp(Gpd)} \simeq \mathrm{Sp}_{\geq 0} \underset{\longleftarrow}{\overset{\pi_0}{\underset{\perp}{\longrightarrow}}} \mathrm{Ab} \simeq \mathrm{CGrp(Set)}$$
In particular both composites $\pi_0\big((-) \otimes \mathbb{S}\big)$ and $\pi_0\big((-) \otimes \mathbb{Z}\big)$ preserve colimits when seen as functors $\mathrm{Sp}_{\geq 0} \to \mathrm{Ab}$.

Since the smallest full subcategory of $\mathrm{Sp}_{\geq 0}$ closed under colimits and containing $\mathbb{S}$ is $\mathrm{Sp}_{\geq 0}$ itself, it follows that the canonical comparison is an isomorphism

$$\pi_0(X) \simeq \pi_0(X \otimes \mathbb{Z})$$

for any connected spectrum X. Equivalently, there is a canonical isomorphism

$$\pi_n(X) \simeq \pi_n(X \otimes \mathbb{Z})$$

for any $(n-1)$-connected spectrum X.

Exercise 1. Since Stiefel–Whitney classes are stable and natural, it suffices to prove the desired formula

$$\mathrm{Sq}^i\big(w_j(p)\big) = \sum_{k=0}^{i} \tbinom{j+k-i-1}{k} w_{j+k}(p) w_{i-k}(p)$$

for any given real vector bundle p. The proof proceeds by induction on the rank n of p.

- The case $n = 0$ is clear, since in this case $w(p) = 1$.
- Assuming that Wu's second formula holds for rank n vector bundles, we establish that it must also hold for p of rank $n+1$. Since we are working with $\mathbb{F}_2$-coefficients, the splitting principle applies and we may assume that p splits as

  $$p \simeq q \oplus \lambda$$

  where λ is a line bundle. Observe that the formula trivially holds when $i > j$ as well as in the edge case $i = j$. Assuming $i < j$, compute

  $$\begin{aligned}\mathrm{Sq}^i\big(w_j(p)\big) &= \mathrm{Sq}^i\big(w_j(q) + w_1(\lambda) w_{j-1}(q)\big) \\ &= \mathrm{Sq}^i\big(w_j(q)\big) + w_1(\lambda)\,\mathrm{Sq}^i\big(w_{j-1}(q)\big) + w_1(\lambda)^2\,\mathrm{Sq}^{i-1}\big(w_{j-1}(q)\big)\end{aligned}$$

  By the induction hypothesis:

  (a) $\mathrm{Sq}^i\big(w_j(q)\big) = \sum_{k=0}^{i} \tbinom{j+k-i-1}{k} w_{j+k}(q) w_{i-k}(q)$

  (b) $w_1(\lambda)\,\mathrm{Sq}^i\big(w_{j-1}(q)\big)$

  $$\begin{aligned} &= w_1(\lambda) \sum_{k=0}^{i} \tbinom{j+k-i-2}{k} w_{j+k-1}(q) w_{i-k}(q) \\ &= w_1(\lambda) \sum_{k=0}^{i} \Big(\tbinom{j+k-i-1}{k} - \tbinom{j+k-i-2}{k-1}\Big) w_{j+k-1}(q) w_{i-k}(q) \\ &= w_1(\lambda) \left(\sum_{k=0}^{i} \tbinom{j+k-i-1}{k} w_{j+k-1}(q) w_{i-k}(q) + \sum_{k=-1}^{i-1} \tbinom{j+k-i-1}{k} w_{j+k}(q) w_{i-k-1}(q) \right) \\ &= w_1(\lambda) \left(\sum_{k=0}^{i} \tbinom{j+k-i-1}{k} w_{j+k-1}(q) w_{i-k}(q) + \sum_{k=0}^{i} \tbinom{j+k-i-1}{k} w_{j+k}(q) w_{i-k-1}(q) \right) \\ &= w_1(\lambda) \sum_{k=0}^{i} \tbinom{j+k-i-1}{k} \big(w_{j+k-1}(q) w_{i-k}(q) + w_{j+k}(q) w_{i-k-1}(q) \big) \end{aligned}$$

  Here, the first step uses that the identity $\binom{m+1}{n} = \binom{m}{n} + \binom{m}{n-1}$ holds for $0 \leq n \leq m$ but also for $m = -1$ and $n = 0$.

$$\text{(c)}\ \ w_1(\lambda)^2 \operatorname{Sq}^{i-1}\big(w_{j-1}(q)\big) = w_1(\lambda)^2 \sum_{k=0}^{i-1} \tbinom{j+k-i-1}{k} w_{j+k-1}(q) w_{i-k-1}(q)$$
$$= w_1(\lambda)^2 \sum_{k=0}^{i} \tbinom{j+k-i-1}{k} w_{j+k-1}(q) w_{i-k-1}(q)$$

Finally:

$$\begin{aligned}\operatorname{Sq}^i\big(w_j(p)\big) &= \sum_{k=0}^{i} \tbinom{j+k-i-1}{k}\big(w_{j+k}(q) + w_1(\lambda) w_{j+k-1}(q)\big)\big(w_{i-k}(q) + w_1(\lambda) w_{i-k-1}(q)\big)\\ &= \sum_{k=0}^{i} \tbinom{j+k-i-1}{k} w_{j+k}(p) w_{i-k}(p)\end{aligned}$$

Exercise 2. Let $\xi \colon B \to \mathrm{BG}$ be an oriented stable spherical fibration. We proved during last exercise session that ξ admits a Thom isomorphism in $\mathbb{Z}$-cohomology. More explicitly, the composite

$$\begin{array}{ccccc}\mathrm{Hom}_{\mathbb{Z}}(B \otimes \mathbb{Z}; \mathbb{Z}) & \xrightarrow{u(\xi)^*} & \mathrm{Hom}_{\mathbb{Z}}(B \otimes \mathrm{M}(\xi); \mathbb{Z}) & \xrightarrow{\Delta_\xi^*} & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{M}(\xi); \mathbb{Z}) \\ \| & & & & \| \\ \mathrm{C}^{-*}(B; \mathbb{Z}) & & \xrightarrow{(-)\cdot u(\xi)} & & \mathrm{C}^{-*}(\mathrm{M}(\xi); \mathbb{Z})\end{array}$$

is an equivalence of coconnective spectra. In particular, the composite

$$\mathrm{M}(\mathrm{Th}(\xi)) \xrightarrow{\Delta_\xi} B \otimes \mathrm{M}(\xi) \xrightarrow{u(\xi)} B \otimes \mathbb{Z}$$

induces an isomorphism on $\mathbb{Z}$-cohomology. Since this is also the case of the unit $\mathrm{M}(\xi) \to \mathrm{M}(\xi) \otimes \mathbb{Z}$, the induced $\mathbb{Z}$-linear map

$$\mathrm{M}(\xi) \longrightarrow \mathrm{M}(\xi) \otimes \mathbb{Z} \longrightarrow B \otimes \mathbb{Z}$$

induces an isomorphism on $\mathbb{Z}$-cohomology as well.

- If B is finite, then
$$\mathrm{M}(\xi) \simeq \operatorname*{colim}_B \xi \simeq \operatorname*{colim}_B \mathbb{S}$$
is as well, and both $\mathrm{M}(\xi) \otimes \mathbb{Z}$ and $B \otimes \mathbb{Z}$ are dualizable. Therefore, the map
$$\begin{array}{ccc}\mathrm{M}(\xi) \otimes \mathbb{Z} & \longrightarrow & B \otimes \mathbb{Z} \\ \| & & \| \\ \mathrm{Hom}_{\mathbb{Z}}\big(\mathrm{Hom}_{\mathbb{Z}}(\mathrm{M}(\xi) \otimes \mathbb{Z}, \mathbb{Z}), \mathbb{Z}\big) & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathbb{Z}}\big(\mathrm{Hom}_{\mathbb{Z}}(B \otimes \mathbb{Z}, \mathbb{Z}), \mathbb{Z}\big)\end{array}$$
is an equivalence.
- In general, B is a filtered colimit of finite groupoids
$$B \simeq \operatorname*{colim}_{i \in I} B_i$$
and
$$\begin{aligned}\mathrm{M}(\xi) &\simeq \operatorname*{colim}_B \xi \\ &\simeq \operatorname*{colim}_{i \in I} \operatorname*{colim}_{B_i} \xi_i \\ &\simeq \operatorname*{colim}_{i \in I} \mathrm{M}(\xi_i)\end{aligned}$$
We conclude by taking a filtered colimit on the maps
$$\mathrm{M}(\xi_i) \otimes \mathbb{Z} \simeq B_i \otimes \mathbb{Z}$$

for $i \in I$.

Conversely if $M(\xi) \otimes \mathbb{Z} \simeq B \otimes \mathbb{Z}$, then $M(\xi) \to B \otimes \mathbb{Z}$ induces an isomorphism on $\mathbb{Z}$-cohomology and Thom isomorphism holds.

Exercise 3. Fix E a commutative ring spectrum.

(1) For $\xi\colon B \to \mathrm{Pic}(\mathbb{S})$ a stable spherical fibration, compute

$$\begin{aligned} \mathrm{C}^{-*}(\mathrm{M}(\xi); E) &\simeq \mathrm{Hom}_{\mathbb{S}}(\mathrm{M}(\xi), E) \\ &\simeq \mathrm{Hom}_E(\mathrm{M}(\xi) \otimes E, E) \\ &\simeq \mathrm{Hom}_E(\operatorname{colim} \xi \otimes E, E) \\ &\simeq \mathrm{Hom}_{\mathrm{Fun}(B, \mathrm{Mod}(E))}(\xi \otimes E, \underline{E}) \end{aligned}$$

and, similarly

$$\begin{aligned} \mathrm{C}^{-*}(B; E) &\simeq \mathrm{Hom}_E(B \otimes E, E) \\ &\simeq \mathrm{Hom}_E(\operatorname{colim} \underline{E}, E) \\ &\simeq \mathrm{Hom}_{\mathrm{Fun}(B, \mathrm{Mod}(E))}(\underline{E}, \underline{E}) \end{aligned}$$

Observe now that the following data are equivalent:

(i) a Thom class, in other words an arrow $u\colon \mathrm{M}(\xi) \to E$ whose restriction $x^*u\colon \mathbb{S} \to E$ along any point $x\colon * \to B$ is a unit of the commutative ring $\pi_0(E)$

(ii) a natural transformation $u\colon \xi \otimes E \to \underline{E}$ between functors $B \to \mathrm{Pic}(E)$

(iii) a trivialization of the composite $\xi \otimes E\colon B \to \mathrm{Pic}(\mathbb{S}) \to \mathrm{Pic}(E)$

In this case, we obtain a Thom isomorphism in E-cohomology

$$\begin{array}{ccc} \mathrm{C}^{-*}(B; E) & \xrightarrow[\simeq]{(-)\cdot u} & \mathrm{C}^{-*}(\mathrm{M}(\xi); E) \\ \| & & \| \\ \mathrm{Hom}(\underline{E}, \underline{E}) & \xrightarrow[\simeq]{u^*} & \mathrm{Hom}(\xi \otimes E, \underline{E}) \end{array}$$

(2) The connected component of $\mathrm{Pic}(E)$ containing E is exactly $\mathrm{BAut}_E(E)$. Since $\mathrm{Aut}_E(E)$ is the subgroupoid of $\mathrm{End}_E(E) \simeq E$ on invertible connected components, it follows that for $n \geq 0$:

$$\pi_{n+1}(\mathrm{Pic}(E)) := \begin{cases} \pi_0(E)^\times & \text{if } n = 0 \\ \pi_n(E) & \text{otherwise} \end{cases}$$

For $\xi\colon B \to \mathrm{Pic}(\mathbb{S})$ a rank 0 spherical fibration, the composite

$$B \xrightarrow{\ \xi\ } \mathrm{Pic}(\mathbb{S}) \xrightarrow{(-)\otimes E} \mathrm{Pic}(E)$$

factors through $\mathrm{BAut}_E(E)$.

- If $E \equiv \mathbb{F}_2$, then all homotopy groups $\pi_{n+1}(\mathrm{Pic}(\mathbb{F}_2))$ vanish and there exists an $\mathbb{F}_2$-oriented Thom class
$$u(\xi)\colon \mathrm{M}(\xi) \to \mathbb{F}_2$$
and therefore a Thom isomorphism with $\mathbb{F}_2$-coefficients.
- If $E \equiv \mathbb{Z}$ and assuming B connected and pointed, the obstruction to the existence of a Thom class is exactly the induced map
$$\pi_1(B) \to \pi_1(\mathrm{Pic}(\mathbb{Z})) \simeq \mathbb{Z}/2\mathbb{Z}$$
vanishing if and only if ξ is orientable, if and only if the first Stiefel–Whitney class $w_1(\xi)$ with $\mathbb{F}_2$-coefficients vanishes.

Exercise 4. In both cases, we show that invertible modules are shifts of the unit.

(1) Let X be an invertible $\mathbb{Z}$-module, with inverse Y. In particular both X and Y are perfect, and thus are represented by bounded complexes of projective (which here are free since we work over $\mathbb{Z}$) modules. Fix two such representatives X and Y in $\mathrm{Ch}_{\geq -m}(\mathbb{Z})$ for some $m \geq 0$.

The classical tensor product $X \otimes Y$ is already derived, since both X and Y are cofibrant for the projective model structure on $\mathrm{Ch}_{\geq -m}(\mathbb{Z})$. Since X and Y are degreewise projective, Künneth formula yields split short exact sequences

$$0 \longrightarrow \bigoplus_{k=-\infty}^{\infty} \mathrm{H}_k(X) \otimes \mathrm{H}_{n-k}(Y) \longrightarrow \mathrm{H}_n(\mathbb{Z}) \longrightarrow \bigoplus_{k=-\infty}^{\infty} \mathrm{Tor}_1\big(\mathrm{H}_k(X), \mathrm{H}_{n-k-1}(Y)\big) \longrightarrow 0$$

for all n, and because $\mathrm{Tor}_1(A, B)$ is always torsion for $\mathbb{Z}$-modules of finite type A and B, the left term at $n = 0$ cannot vanish. Therefore, there exists r such that $\mathrm{H}_r(X) \otimes \mathrm{H}_{-r}(Y) \simeq \mathbb{Z}$, and this implies in turn

$$\mathrm{H}_r(X) \simeq \mathrm{H}_{-r}(Y) \simeq \mathbb{Z}$$

As an immediate consequence, we obtain $\mathrm{H}_{n-r}(Y) \simeq 0$ and $\mathrm{H}_{n+r}(X) \simeq 0$ for $n \neq 0$, so that $X \simeq \mathbb{Z}[r]$ and $Y \simeq \mathbb{Z}[-r]$.

(2) Let X be an inversible $\mathbb{S}$-module, and assume without loss of generality that $X \otimes \mathbb{Z} \simeq \mathbb{Z}$. Since X is dualizable, it is $(-m)$-connective for some $m \geq 0$, and stable Hurewicz then implies that X is connective and yields an identification

$$\pi_0(X) \simeq \pi_0(X \otimes \mathbb{Z}) \simeq \mathbb{Z}$$

Since the first map is obtained by the composition

$$\begin{aligned} \pi_0(X) &\simeq \pi_0 \operatorname{Hom}_{\mathbb{S}}(\mathbb{S}, X) \\ &\to \pi_0 \operatorname{Hom}_{\mathbb{S}}(\mathbb{S}, X \otimes \mathbb{Z}) \\ &\simeq \pi_0 \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, X \otimes \mathbb{Z}) \\ &\simeq \pi_0(X \otimes \mathbb{Z}) \end{aligned}$$

there exists a morphism $\alpha \colon \mathbb{S} \to X$ inducing an equivalence after applying $(-) \otimes \mathbb{Z}$. In particular

$$\begin{aligned} (\operatorname{cofib} \alpha) \otimes \mathbb{Z} &\simeq \operatorname{cofib} \alpha \otimes \mathbb{Z} \\ &\simeq 0 \end{aligned}$$

Both $\mathbb{S}$ and X are connective so $\operatorname{cofib} \alpha$ is connective as well, and stable Hurewicz then implies $\operatorname{cofib} \alpha \simeq 0$. Finally, α is an equivalence $\mathbb{S} \simeq X$.