TOPOLOGIE IV – EXERCISE SHEET 5

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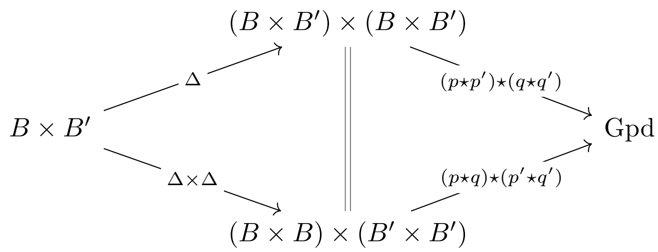
Joins distribute over sums. Consider two groupoids B and B' together with spherical fibrations

$$p, q\colon B \to \mathrm{Gpd} \quad \text{and} \quad p', q'\colon B' \to \mathrm{Gpd}$$

Since the functors $(-)\times(-)$ and $(-)\star(-)$ are both associative and commutative up to homotopy as functors $\mathrm{Gpd}\times\mathrm{Gpd}\to\mathrm{Gpd}$, we obtain a commutative diagram



and in particular a canonical identification

$$(p \star p') \oplus (q' \star q') \simeq (p \oplus q) \star (p' \oplus q')$$

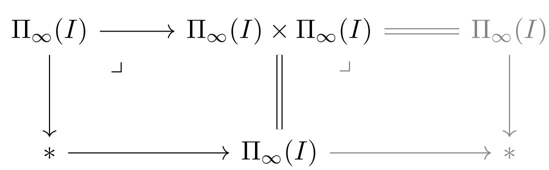
As a consequence:

$$(p \star p')^{-1} \simeq p^{-1} \star (p')^{-1}$$

Sifted categories are weakly contractible. If I is a sifted category, for instance filtered, then:

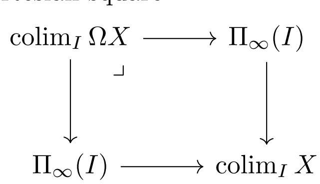
$$\begin{aligned}\Pi_\infty(I) &\simeq \operatorname*{colim}_I * \\ &\simeq \operatorname*{colim}_{I\times I} * \\ &\simeq \Pi_\infty(I)\times\Pi_\infty(I)\end{aligned}$$

so that the diagonal map $\Pi_\infty(I) \to \Pi_\infty(I)\times\Pi_\infty(I)$ is an equivalence. In particular either projection $\Pi_\infty(I)\times\Pi_\infty(I) \to \Pi_\infty(I)$ must also be invertible. Since I is not empty by assumption, we can paste cartesian squares



and $\Pi_\infty(I) \simeq *$.

Loops and filtered colimits. Since limits and weakly contractible colimits in Gpd_* are computed in Gpd, forming pullbacks in Gpd_* commutes with all filtered colimits. In particular, given a filtered diagram $X\colon I \to \mathrm{Gpd}_*$, we have the following cartesian square



Since $\Pi_\infty(I) \simeq *$, the canonical comparison map yields an isomorphism

$$\operatorname*{colim}_I \Omega X \simeq \Omega\big(\operatorname*{colim}_I X\big)$$

In other words, the endofunctor $\Omega\colon \mathrm{Gpd}_* \to \mathrm{Gpd}_*$ preserves filtered colimits.

Inverting an endofunctor. Let $\mathcal{C}$ a category with sequential colimits, equipped with an endofunctor T preserving sequential colimits and a natural transformation

$$\alpha\colon \mathrm{id}_{\mathcal{C}} \to T$$

satisfying $\alpha T \simeq T\alpha$ (this condition is not automatic, as one can see on the free abelian group monad on Set for instance). The full subcategory $i\colon \mathcal{C}_\alpha \subseteq \mathcal{C}$ on those objects x such that $\alpha(x)\colon x \to T(x)$ is an isomorphism is reflexive, with left adjoint given by

$$L :\simeq \operatorname{colim}\big(\mathrm{id}_{\mathcal{C}} \xrightarrow{\alpha} T \xrightarrow{\alpha} T^2 \xrightarrow{\alpha} \cdots\big)$$

Proof. Since T commutes with filtered colimits and $\alpha T \simeq T\alpha$, the functor $L\colon \mathcal{C} \to \mathcal{C}$ indeed factors through $\mathcal{C}_\alpha$ and we have a canonical identification $\varepsilon\colon Li \simeq \mathrm{id}_{\mathcal{C}_\alpha}$. Define

$$\eta\colon \mathrm{id}_{\mathcal{C}} \to iL$$

be the structural inclusion into the colimit. It remains to show that the triangular identities

$$i \xrightarrow[\eta i]{} iLi \xrightarrow[i\varepsilon]{} i \quad (\text{composite } \mathrm{id}_i) \qquad \text{and} \qquad L \xrightarrow[L\eta]{} LiL \xrightarrow[\varepsilon L]{} L \quad (\text{composite } \mathrm{id}_L)$$

are satified. The left one is clear, so let us focus on the right one. By definition of η, the natural transformation

$$L\eta\colon \operatorname*{colim}_m T^m \longrightarrow \operatorname*{colim}_{m,n} T^{m+n}$$

is induced by the structural maps $T^m \longrightarrow \operatorname{colim}_n T^{m+n}$ for $m \geq 0$. Also by definition of ε, the composite

$$\operatorname{colim}_m T^m \longrightarrow \operatorname{colim}_{m,n} T^{m+n} \xrightarrow{\varepsilon L} \operatorname{colim}_m T^m$$

is the identity, where the first map is the inclusion into the colimit at induced by $m \mapsto (m, 0)$. This is exactly the desired triangular identity. ☐

Exercise 1. Let B and B' be two groupoids. Observe that for two completed $\mathbb{F}_2$-cohomology classes $x \in \mathrm{H}^*(B;\mathbb{F}_2)^\wedge$ and $y \in \mathrm{H}^*(B';\mathbb{F}_2)^\wedge$ one has:

$$\begin{aligned}\mathrm{Sq}(\mathrm{Sq}^{-1}(x) \times \mathrm{Sq}^{-1}(y)) &= \mathrm{Sq}\,\mathrm{Sq}^{-1}(x) \times \mathrm{Sq}\,\mathrm{Sq}^{-1}(y)\\ &= x \times y\end{aligned}$$

in $\mathrm{H}^*(B \times B';\mathbb{F}_2)^\wedge$. In other words, Sq^{-1} satisfies the Cartan formula

$$\mathrm{Sq}^{-1}(x \times y) = \mathrm{Sq}^{-1}(x) \times \mathrm{Sq}^{-1}(y)$$

and from there the proof that Wu classes satisfy Cartan formula is the same as for Stiefel–Whitney classes.

Namely, if p and p' are spherical fibrations over B and B' respectively, then

$$v(p \star p') = v(p) \times v(p')$$

Furthermore, if $B \equiv B'$:

$$v(p \oplus p') = v(p)\ \cdot\ v(p')$$

by pulling the previous formula along the diagonal $B \to B \times B$.

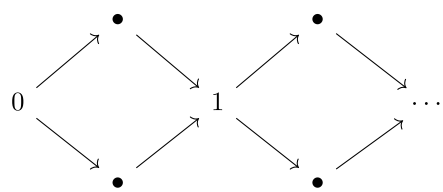
We can alternatively prove these Cartan formulas using Wu's first formula and the Cartan formula for Stiefel–Whitney classes. Indeed, one has

$$\mathrm{Sq}\left(v(p \star p')\right) = w\big((p \star p')^{-1}\big)$$

$$\begin{aligned} &= w\big(p^{-1} \star (p')^{-1}\big) \\ &= w\big(p^{-1}\big) \times w\big((p')^{-1}\big) \\ &= \mathrm{Sq}\,\big(v(p)\big) \times \mathrm{Sq}\,\big(v(p')\big) \\ &= \mathrm{Sq}\,\big(v(p) \times v(p')\big) \end{aligned}$$

in $\mathrm{H}^*(B \times B'; \mathbb{F}_2)^\wedge$, and it then suffices to apply Sq^{-1} on both sides.

Exercise 2. We begin by computing Hom groupoids in PSp, by observing that PSp is equivalently described as the full subcategory of diagrams in Gpd_* indexed by



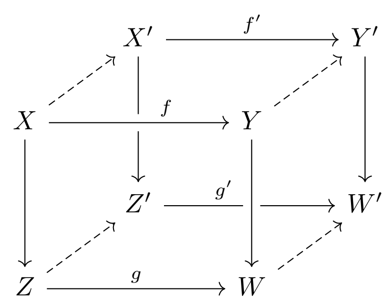
such that each $\bullet$ is sent to $*$. Denoting this indexing poset by I, the decomposition

$$I \simeq I_{0//1} \amalg_1 I_{1//2} \amalg_2 \cdots$$

yields

$$\mathrm{Fun}(I, \mathrm{Gpd}_*) \simeq \mathrm{Fun}(I_{0//1}, \mathrm{Gpd}_*) \times_{\mathrm{Gpd}_*} \mathrm{Fun}(I_{1//2}, \mathrm{Gpd}_*) \times_{\mathrm{Gpd}_*} \cdots$$

For all $i \geq 0$, notice that $I_{i//i+1} \simeq [1] \times [1]$. But by what was done in the correction to exercise sheet 1, the space of morphisms between two commutative squares in Gpd_*



is computed as

$$\mathrm{Hom}_{[1]\times[1]}\left(\square, \square'\right) \simeq \mathrm{Hom}_{[1]}(f, f') \times_{\mathrm{Hom}_{[1]}(f,g')} \mathrm{Hom}_{[1]}(g, g')$$

and so is the limit of the following diagram

$$\begin{array}{ccccc} \mathrm{Hom}_*(X, X') & \longrightarrow & \mathrm{Hom}_*(X, Y') & \longleftarrow & \mathrm{Hom}_*(Y, Y') \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Hom}_*(X, Z') & \longrightarrow & \mathrm{Hom}_*(X, W') & \longleftarrow & \mathrm{Hom}_*(Y, W') \\ \uparrow & & \uparrow & & \uparrow \\ \mathrm{Hom}_*(Z, Z') & \longrightarrow & \mathrm{Hom}_*(Z, W') & \longleftarrow & \mathrm{Hom}_*(W, W') \end{array}$$

When Y, Y', Z and Z' are contractible, this limit simplifies to

$$\mathrm{Hom}_{[1]\times[1]}\left(\square, \square'\right) \simeq \mathrm{Hom}_*(X, X') \times_{\mathrm{Hom}_*(X, \Omega W')} \mathrm{Hom}_*(W, W')$$

In particular, the canonical map

$$\mathrm{oplaxlim}\left(\mathrm{Gpd}_* \xleftarrow{\Omega} \mathrm{Gpd}_*\right) \to \mathrm{Fun}([1] \times [1], \mathrm{Gpd}_*)$$

is fully faithful, and its essential image consists exactly of commutative squares of the form

$$\begin{array}{ccc} X & \longrightarrow & * \\ \big\downarrow & & \big\downarrow \\ * & \longrightarrow & Y \end{array}$$

Finally, this gives an identification

$$\mathrm{PSp} \simeq \mathrm{oplaxlim}\left(\mathrm{Gpd}_* \xleftarrow{\Omega} \mathrm{Gpd}_* \xleftarrow{\Omega} \cdots \right)$$

More concretely, the ∞-category PSp identifies with the category of sequences $(X_n)_{n\geq 0}$ of pointed groupoids equipped with maps $X_n \to \Omega X_{n+1}$. In particular the Hom groupoid $\mathrm{Hom}_{\mathrm{PSp}}(X, Y)$ between two prespectra X and Y is computed as the limit of the following diagram

$$\begin{array}{ccccccccc} \mathrm{Hom}_*(X_0, Y_0) & & & & \cdots & & \mathrm{Hom}_*(X_n, Y_n) & & \cdots \\ & \searrow & & \swarrow & & \searrow\!\cdots\searrow & & \swarrow \quad \searrow & & \swarrow \\ & & \mathrm{Hom}_*(X_0, \Omega Y_1) & & & \mathrm{Hom}_*(X_{n-1}, \Omega Y_n) & & \mathrm{Hom}_*(X_n, \Omega Y_{n+1}) & \end{array}$$

We are now ready to compute the required adjoints:

(1) Consider the functor

$$\Sigma^\infty_{\mathrm{PSp}} \colon \mathrm{Gpd}_* \to \mathrm{PSp}$$

given by the following oplax cone:

$$\begin{array}{ccccccccc} \mathrm{Gpd}_* & = & \cdots & = & \mathrm{Gpd}_* & = & \mathrm{Gpd}_* & = & \cdots \\ \big\downarrow{\scriptstyle \mathrm{id}} & \overset{\eta}{\Searrow} & \big\downarrow{\scriptstyle \cdots} & \overset{\eta\Sigma^{n-1}}{\Searrow} & \big\downarrow{\scriptstyle \Sigma^n} & \overset{\eta\Sigma^{n}}{\Searrow} & \big\downarrow{\scriptstyle \Sigma^{n+1}} & \Searrow & \\ \mathrm{Gpd}_* & \xleftarrow{\Omega} & \cdots & \xleftarrow{\Omega} & \mathrm{Gpd}_* & \xleftarrow{\Omega} & \mathrm{Gpd}_* & \xleftarrow{\Omega} & \cdots \end{array}$$

For X a pointed groupoid and Y a prespectra, the Hom groupoid from $\Sigma^\infty_{\mathrm{PSp}} X$ to Y is computed by the limit of

$$\begin{array}{ccccccccc} \mathrm{Hom}_*(X, Y_0) & & & & \cdots & & \mathrm{Hom}_*(\Sigma^n X, Y_n) & & \cdots \\ & \searrow & & /\!\!/ & & \searrow\!\cdots\searrow & & /\!\!/ \quad \searrow & & /\!\!/ \\ & & \mathrm{Hom}_*(X, \Omega Y_1) & & & \mathrm{Hom}_*(\Sigma^{n-1} X, \Omega Y_n) & & \mathrm{Hom}_*(\Sigma^n X, \Omega Y_{n+1}) & \end{array}$$

and therefore:

$$\begin{aligned} \mathrm{Hom}_{\mathrm{PSp}}(\Sigma^\infty_{\mathrm{PSp}} X, Y) &\simeq \lim_{n\in\mathbb{N}} \mathrm{Hom}_*(\Sigma^n X, Y_n) \\ &\simeq \mathrm{Hom}_*(X, Y_0) \end{aligned}$$

because the category $\mathbb{N}$ has an initial object. We thus have a Bousfield colocalization

$$\mathrm{Gpd}_* \underset{\mathrm{ev}_0}{\overset{\Sigma^\infty_{\mathrm{PSp}}}{\rightleftarrows}} \mathrm{PSp} \qquad (\perp)$$

since the unit is the structural identification $\mathrm{id}_{\mathrm{Gpd}_*} \simeq \mathrm{ev}_0 \circ \Sigma^\infty_{\mathrm{PSp}}$.

(2) Name δ_n the structural natural transformation $\mathrm{ev}_n \to \Omega \mathrm{ev}_{n+1}$ between functors $\mathrm{PSp} \to \mathrm{Gpd}_*$. The commutative diagram

$$\begin{array}{ccccccccc}
\mathrm{PSp} & = & \cdots & = & \mathrm{PSp} & = & \mathrm{PSp} & = & \cdots \\
\downarrow{\scriptstyle \Omega \mathrm{ev}_1} & \overset{\Omega\delta_1}{\searrow} & \downarrow{\scriptstyle \cdots} & \overset{\Omega\delta_n}{\searrow} & \downarrow{\scriptstyle \Omega \mathrm{ev}_{n+1}} & \overset{\Omega\delta_{n+1}}{\searrow} & \downarrow{\scriptstyle \Omega \mathrm{ev}_{n+2}} & \searrow & \\
\mathrm{Gpd}_* & \xleftarrow{\Omega} & \cdots & \xleftarrow{\Omega} & \mathrm{Gpd}_* & \xleftarrow{\Omega} & \mathrm{Gpd}_* & \xleftarrow{\Omega} & \cdots
\end{array}$$

defines a translation endofunctor

$$T\colon \mathrm{PSp} \to \mathrm{PSp}$$

Since $\mathrm{id}\colon \mathrm{PSp} \to \mathrm{PSp}$ is induced by the diagram

$$\begin{array}{ccccccccc}
\mathrm{PSp} & = & \cdots & = & \mathrm{PSp} & = & \mathrm{PSp} & = & \cdots \\
\downarrow{\scriptstyle \mathrm{ev}_0} & \overset{\delta_0}{\searrow} & \downarrow{\scriptstyle \cdots} & \overset{\delta_{n-1}}{\searrow} & \downarrow{\scriptstyle \mathrm{ev}_{n}} & \overset{\delta_{n}}{\searrow} & \downarrow{\scriptstyle \mathrm{ev}_{n+1}} & \searrow & \\
\mathrm{Gpd}_* & \xleftarrow{\Omega} & \cdots & \xleftarrow{\Omega} & \mathrm{Gpd}_* & \xleftarrow{\Omega} & \mathrm{Gpd}_* & \xleftarrow{\Omega} & \cdots
\end{array}$$

one can define by the universal property of PSp a natural transformation

$$\delta\colon \mathrm{id}_{\mathrm{PSp}} \to T$$

using the natural transformations δ_n on each projection. Indeed, the naturality is exactly the data of commutative squares

$$\begin{array}{ccc}
\mathrm{ev}_n & \xrightarrow{\delta_n} & \Omega\,\mathrm{ev}_{n+1} \\
\downarrow{\scriptstyle \delta_n} & & \downarrow{\scriptstyle \Omega\delta_{n+1}} \\
\Omega\,\mathrm{ev}_{n+1} & \xrightarrow{\Omega\delta_{n+1}} & \Omega^2\,\mathrm{ev}_{n+2}
\end{array}$$

for $n \geq 0$, which we fill by the identity homotopy.

Heuristically, T sends a sequence $(X_n)_{n\geq 0}$ to the sequence $\left(\Omega X_{n+1}\right)_{n\geq 0}$, and for $n \geq 0$ the map $\delta(X)_n$ is the structural morphism $X_n \to \Omega X_{n+1}$. In particular the fullsubategory $\mathrm{PSp}_\delta \subset \mathrm{PSp}$ on those prespectra X such that $\delta(X)$ is an isomorphism identifies with Sp.

(a) The natural transformations $\mathrm{ev}_n \circ \delta T$ and $\mathrm{ev}_n \circ T\delta$ are by definition both computed as

$$\Omega\delta_{n+1}\colon \Omega\,\mathrm{ev}_{n+1} \to \Omega^2\,\mathrm{ev}_{n+2}$$

for $n \geq 0$, so that one can construct an homotopy $\delta T \simeq T\delta$.

(b) Since the inclusion $\mathrm{PSp} \subset \mathrm{Fun}(I, \mathrm{Gpd}_*)$ preserves and reflects weakly contractible colimits, this is in particular the case for filtered colimits. Since the endofunctor $\Omega\colon \mathrm{Gpd}_* \to \mathrm{Gpd}_*$ commutes with filtered colimits, we conclude that T also commutes with filtered colimits.

In particular, the above discussion applies, and we obtain a Bousfield localization

$$\mathrm{PSp} \underset{\perp}{\overset{L}{\rightleftarrows}} \mathrm{Sp}$$

Because Ω commutes with filtered colimits, the right adjoint is ω-accessible and thus Sp is compactly generated[1]. We also obtain that the *spectrification* functor L can be computed as the following colimit in PSp:

$$L \simeq \mathrm{colim}\left(\mathrm{id}_{\mathrm{PSp}} \xrightarrow{\delta} T \xrightarrow{\delta} T^2 \xrightarrow{\delta} \cdots\right)$$

In other words, we have

$$LX_n \simeq \operatorname*{colim}_k \Omega^k X_{n+k}$$

for any prespectrum X and $n \geq 0$. Composing adjunctions, we finally obtain

[1] Here, we use that the inclusion $\mathrm{PSp} \subset \mathrm{Fun}(I, \mathrm{Gpd}_*)$ preserves weakly contractible colimits and so is ω-accessible, and that $\mathrm{Fun}(I, \mathrm{Gpd}_*)$ is compactly generated.

$$\mathrm{Gpd}_* \underset{\Omega^\infty}{\overset{\Sigma^\infty}{\rightleftarrows}} \mathrm{Sp} \quad (\perp)$$

where the right adjoint preserves filtered colimits. This is neither a localization nor a colocalization.

(3) Since the unit map $X \to LX$ is simply the pointwise inclusion

$$X_n \to \operatorname*{colim}_k \Omega^k X_{n+k}$$

into the colimit, it induces a chain of identifications

$$\begin{aligned}\pi_*(X) &\simeq \operatorname*{colim}_n \pi_{*+n}(X_n)\\ &\simeq \operatorname*{colim}_{n+k} \pi_{*+n+k}(X_{n+k})\\ &\simeq \operatorname*{colim}_n \pi_{*+n}(LX_n)\\ &\simeq \pi_*(LX)\end{aligned}$$

For $f\colon X \to Y$ a map of prespectra, the commutative diagram

$$\begin{array}{ccc}\pi_*(X) & \xrightarrow{f_*} & \pi_*(Y)\\ \| & & \|\\ \pi_*(LX) & \xrightarrow{Lf_*} & \pi_*(LY)\end{array}$$

shows that f induces an isomorphism on π_* if and only if Lf does, and this is the case if and only if Lf is an equivalence since LX and LY are spectra. In particular L inverts exactly those maps that are sent by π_* to equivalences.

(4) Since both functors

$$\mathrm{Sp} \hookrightarrow \mathrm{PSp} \hookrightarrow \mathrm{Fun}(I, \mathrm{Gpd}_*)$$

preserve limits, it follows that $\Omega\colon \mathrm{Sp} \to \mathrm{Sp}$ is computed pointwise either in PSp or in $\mathrm{Fun}(I, \mathrm{Gpd}_*)$. Therefore

$$\begin{aligned}\Omega(X)_{n+1} &\simeq \Omega X_{n+1}\\ &\simeq X_n\end{aligned}$$

for any spectrum X, and Ω sends a spectrum $X_0, X_1, X_2, \ldots$ to the shifted sequence $\Omega X_0, X_0, X_1, \ldots$.

(5) For X a compact object in Gpd_* and Y another pointed groupoid:

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Sp}}(\Sigma^\infty X, \Sigma^\infty Y) &\simeq \mathrm{Hom}_*(X, \Omega^\infty\Sigma^\infty Y)\\ &\simeq \mathrm{Hom}_*\big(X, \operatorname*{colim}_n \Omega^n\Sigma^n Y\big)\\ &\simeq \operatorname*{colim}_n \mathrm{Hom}_*(\Sigma^n X, \Sigma^n Y)\end{aligned}$$

Since S^0 is compact, we obtain

$$\mathrm{End}_{\mathrm{Sp}}(\mathbb{S}) \simeq \operatorname*{colim}_n \mathrm{End}_*(S^n)$$

But $\mathrm{Aut}_*(S^n) \xrightarrow{\Sigma} \mathrm{Aut}_*(S^{n+1})$ is an isomorphism on connected components for all n, and therefore

$$\begin{aligned}\mathrm{Aut}_{\mathrm{Sp}}(\mathbb{S}) &\simeq \operatorname*{colim}_n \mathrm{Aut}_*(S^n)\\ &\simeq \operatorname*{colim}_n \mathrm{G}(n)\\ &\simeq \mathrm{G}\end{aligned}$$

and BG identifies with the subcategory $\mathrm{BAut}_{\mathrm{Sp}}(\mathbb{S})$ of Sp. Furthermore, the proof also shows that the following squares

$$\begin{CD} \mathrm{BG}(n) @>>> \mathrm{BG} \\ @VVV @VVV \\ \mathrm{Gpd}_* @>{\Sigma^{\infty-n}}>> \mathrm{Sp} \end{CD}$$

commute for $n \geq 0$, where the two vertical maps are subcategory inclusions.

Exercise 3. Fix a stable spherical fibration $\xi \colon B \to \mathrm{BG}$.

(1) For $n \geq 0$, consider the cartesian square

$$\begin{CD} B_n @>{i}>> B_{n+1} @>>> B \\ @V{\xi_n}VV @V{\xi_{n+1}}VV @VVV \\ \mathrm{BG}(n) @>{\Sigma}>> \mathrm{BG}(n+1) @>>> \mathrm{BG} \end{CD}$$

The homotopy $\Sigma\xi_n \simeq i^*\xi_{n+1}$ induces a pointed map

$$\Sigma\mathrm{Th}(\xi_n) \simeq \mathrm{Th}(\Sigma\xi_n) \to \mathrm{Th}(\xi_{n+1})$$

or equivalently

$$\mathrm{Th}(\xi_n) \to \Omega\mathrm{Th}(\xi_{n+1})$$

The resulting prespectrum is denoted $\mathrm{Th}(\xi)$. Writing the Thom space as a colimit, this construction can be made functorial in ξ so that we actually have a functor $\mathrm{Th}\colon \mathrm{Gpd}_{/\mathrm{BG}} \to \mathrm{PSp}$.

(2) To prove the colimit formula for $\mathrm{M}(\xi) :\simeq L\big(\mathrm{Th}(\xi)\big)$, we distinguish two cases

- If ξ factorizes as
$$B \xrightarrow{\xi_n} \mathrm{BG}(n) \xrightarrow{\Sigma^{\infty-n}} \mathrm{BG}$$
for some n, then
$$\begin{aligned} \mathrm{M}(\xi) &\simeq \Sigma^{\infty-n}\,\mathrm{Th}(\xi_n) \\ &\simeq \Sigma^{\infty-n} \operatorname*{colim}_B \xi_n \\ &\simeq \operatorname*{colim}_B \Sigma^{\infty-n}\,\xi_n \\ &\simeq \operatorname*{colim}_B \xi \end{aligned}$$
since $\Sigma^{\infty-n} :\simeq \Omega^n\Sigma^\infty$ is cocontinuous.

- In general, then observe that
$$\mathrm{Th}(\xi) \simeq \operatorname*{colim}_n \mathrm{Th}(\Sigma^{\infty-n}\,\xi_n)$$
Indeed, filtered colimits are computed pointwise in PSp, and this diagram is pointwise eventually constant. Using now that L is a left adjoint:
$$\begin{aligned} \mathrm{M}(\xi) &\simeq \operatorname*{colim}_n L\big(\mathrm{Th}(\Sigma^{\infty-n}\,\xi_n)\big) \\ &\simeq \operatorname*{colim}_n \operatorname*{colim}_{B_n} \Sigma^{\infty-n}\,\xi_n \\ &\simeq \operatorname*{colim}_{\operatorname{colim}_n B_n} \xi \\ &\simeq \operatorname*{colim}_B \xi \end{aligned}$$
The last step uses that $B \simeq \operatorname{colim}_n B_n$, but this is a consequence of $\mathrm{BG} \simeq \operatorname{colim}_n \mathrm{BG}(n)$ and of the universality of weakly contractible colimits in Gpd_*.

(3) Let $\mathcal{C}$ denote any category with B-colimits, and fix a functor $F\colon B \to \mathcal{C}$. Then the diagonal map $B \to B \times B$ induces a morphism in $\mathcal{C}$:

$$\operatorname*{colim}_B F \to \operatorname*{colim}_{B\times B} F \circ \mathrm{pr}_2 \simeq B \otimes \operatorname*{colim}_B F$$

Specializing to our situation, we have a well defined diagonal map

$$\Delta_\xi \colon \mathrm{M}(\xi) \to B \otimes \mathrm{M}(\xi)$$

in Sp.

Exercise 4. Fix an oriented stable fibration $\xi\colon B \to \mathrm{BG}$.

(1) By adjunction

$$\begin{aligned}\mathrm{H}^0(\mathrm{M}(\xi)) &\simeq \pi_0 \operatorname{Hom}_{\mathrm{Sp}}(\mathrm{M}(\xi), \mathbb{Z}) \\ &\simeq \pi_0 \operatorname{Hom}_{\mathrm{PSp}}(\mathrm{Th}(\xi), \mathbb{Z})\end{aligned}$$

For $n \geq 0$, the following diagram commutes

$$\begin{array}{ccc} \mathrm{Th}(\xi_n) & \longrightarrow & \Omega\mathrm{Th}(\xi_{n+1}) \\ \downarrow{\scriptstyle u(\xi_n)} & & \downarrow{\scriptstyle u(\xi_{n+1})} \\ \mathrm{K}(\mathbb{Z}, n) & = & \Omega\mathrm{K}(\mathbb{Z}, n+1) \end{array}$$

Indeed, this is evident if B_n is empty, and if not, consider the restrictions

$$S^{n+1} \simeq \Sigma\mathrm{Th}(S^{n-1} \to *) \longrightarrow \Sigma\mathrm{Th}(\xi_n) \longrightarrow \mathrm{Th}(\xi_{n+1})$$

induced by the inclusion of any point $* \to B_n$. The classes $u(\xi_n)$ therefore assemble into a map

$$u(\xi)\colon \mathrm{M}(\xi) \to \mathbb{Z}$$

(2) Given a point $x\colon * \to B$, consider the following diagram

$$\begin{array}{ccccc} \mathrm{Th}(\xi(x)) & \longrightarrow & \mathrm{M}(\xi(x)) & = & \mathbb{S} \\ \downarrow & & \downarrow{\scriptstyle \mathrm{M}(i)} & & \downarrow \\ \mathrm{Th}(\xi) & \longrightarrow & \mathrm{M}(\xi) & \xrightarrow{u(\xi)} & \mathbb{Z} \end{array}$$

Since restriction along the left most map sends each $u(\xi_n)$ on $\left[S^n\right]$ as soon as B_n contains x, it follows that

$$\mathrm{M}(i)^* u(\xi) \simeq 1$$

in $\mathbb{S}$.

(3) We distinguish between two cases.

- Assume first that ξ factors through $\mathrm{BG}(n)$ for some $n \geq 1$, and compute for $k \geq 0$:

$$\begin{aligned}\mathrm{H}^k(\mathrm{M}(\xi);\mathbb{Z}) &\simeq \pi_0 \operatorname{Hom}_{\mathrm{PSp}}\big(\mathrm{Th}(\xi), \Sigma^k\mathbb{Z}\big) \\ &\simeq \pi_0\Big(\lim_{m\geq n} \operatorname{Hom}_*\big(\mathrm{Th}(\xi_m), \mathrm{K}(\mathbb{Z}, m+k)\big)\Big) \\ &\to \lim_{m\geq n} \pi_0 \operatorname{Hom}_*\big(\Sigma^{m-n}\mathrm{Th}(\xi_n), \mathrm{K}(\mathbb{Z}, m+k)\big)\end{aligned}$$

  where the second lines uses the computation of Hom groupoids in PSp and the fact that $\mathbb{Z}$ is a spectrum. But by Thom isomorphism, the last sequential limit is constant on $\mathrm{H}^k(B;\mathbb{Z})$, so that the last map admits an inverse. Finally one obtains an identification

$$\mathrm{H}^k(B;\mathbb{Z}) \simeq \mathrm{H}^k(\mathrm{M}(\xi);\mathbb{Z})$$

  More explicitly, it is given by taking the limit of the following (eventually constant) cone

$$\begin{array}{ccccccc}
\mathrm{H}^k(B;\mathbb{Z}) & = & \cdots & = & \mathrm{H}^k(B;\mathbb{Z}) & = & \cdots \\
\downarrow{\scriptstyle (-)\cdot u(\xi_n)} & & \downarrow{\scriptstyle \cdots} & & \downarrow{\scriptstyle (-)\cdot u(\xi_m)} & & \\
\mathrm{H}^{n+k}(\mathrm{Th}(\xi_n);\mathbb{Z}) & = & \cdots & = & \mathrm{H}^{m+k}\big(\mathrm{Th}(\xi_m);\mathbb{Z}\big) & = & \cdots
\end{array}$$

Unwinding definitions, it is given by applying $\pi_{-k}(-)$ to the following composite

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathbb{Z}}(B\otimes\mathbb{Z},\mathbb{Z}) & \xrightarrow{u(\xi)^*} & \mathrm{Hom}_{\mathbb{Z}}(B\otimes \mathrm{M}(\xi),\mathbb{Z}) & \xrightarrow{\Delta_\xi^*} & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{M}(\xi),\mathbb{Z}) \\
\| & & & & \| \\
\mathrm{C}^{-*}(B;\mathbb{Z}) & & \xrightarrow{\qquad (-)\cdot u(\xi) \qquad} & & \mathrm{C}^{-*}(\mathrm{M}(\xi);\mathbb{Z})
\end{array}$$

which must therefore be an equivalence between coconnective spectra.

- In general, the first case implies that we have equivalences

$$(-)\cdot u(\xi_n)\colon \mathrm{C}^{-*}(B_n;\mathbb{Z}) \simeq \mathrm{C}^{-*}\big(\mathrm{M}(\Sigma^{\infty-n}\xi_n);\mathbb{Z}\big)$$

natural in n. Taking limits on both sides shows that

$$(-)\cdot u(\xi)\colon \mathrm{C}^{-*}(B;\mathbb{Z}) \simeq \mathrm{C}^{-*}(\mathrm{M}(\xi);\mathbb{Z})$$

and this is exactly Thom isomorphism.