TOPOLOGIE IV – EXERCISE SHEET 4

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Quaternionification. The division $\mathbb{R}$-algebra $\mathbb{H}$ is canonically a $(\mathbb{C}, \mathbb{H})$-bimodule, so that $V \otimes_{\mathbb{C}} \mathbb{H}$ inherits a canonical structure of right $\mathbb{H}$-module for any $\mathbb{C}$-module V. Observe that, because $ij = -ji$ in $\mathbb{H}$, the formula

$$v \otimes (x + jy) \mapsto (vx, vy)$$

defines a $(\mathbb{C}, \mathbb{C})$-linear isomorphism $V \otimes_{\mathbb{C}} \mathbb{H} \simeq V \oplus \overline{V}$, where $\overline{V} :\simeq V \otimes_{\mathbb{C}} \overline{\mathbb{C}}$.

Monoid actions and modules. Fix a monoid object M in Gpd, and let $(\mathcal{C}, \otimes, \mathbb{1})$ be a presentably symmetric monoidal category, or in other terms an commutative algebra in Pr^{L}. In particular $\mathcal{C}$ receives a unique symmetric monoidal functor from the unit

$$\mathbb{1} \otimes (-) \colon \mathrm{Gpd} \to \mathcal{C}$$

and $\mathbb{1}[M] :\simeq \mathbb{1} \otimes M$ inherits a canonical algebra structure from M. Under our assumptions, the adjunction

$$\begin{tikzcd} \mathcal{C} \simeq \mathrm{LMod}(\mathbb{1}) \arrow[r, bend left, "(-)\otimes\mathbb{1}[M]"] \arrow[r, phantom, "\perp"] & \mathrm{LMod}(\mathbb{1}[M]) \arrow[l, bend left, "\text{forget}"] \end{tikzcd}$$

exists and is monadic, so that

$$\mathrm{LMod}(\mathbb{1}[M]) \simeq \mathrm{Alg}((-) \otimes \mathbb{1}[M])$$

where $(-) \otimes \mathbb{1}[M] \colon \mathcal{C} \to \mathcal{C}$ denotes the monad induced by the adjunction.

On the other hand, if $\mathrm{B}M$ denotes the classifying category of M, then the adjunction

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathrm{Fun}(\mathrm{B}M, \mathcal{C}) \arrow[l, bend left, "\mathrm{ev}_*"] \end{tikzcd}$$

is again monadic and one identifies the left adjoint with $(-) \otimes M$ using the pointwise formula for left Kan extensions and pasting cartesian squares

$$\begin{tikzcd} M \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{B}M_{/*} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Mor}(\mathrm{B}M) \arrow[d] \\ * \arrow[r] & \mathrm{B}M \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{B}M \times \mathrm{B}M \arrow[d] \\ & * \arrow[r] & \mathrm{B}M \end{tikzcd}$$

But observe that

$$\begin{aligned} (-) \otimes \mathbb{1}[M] &\simeq (-) \otimes (\mathbb{1} \otimes M) \\ &\simeq (-) \otimes M \end{aligned}$$

and this is an isomorphism of monads since the algebra structure of $\mathbb{1}[M]$ is inherited from M.

In particular we have a canonical identification

$$\mathrm{Fun}(\mathrm{B}M, \mathcal{C}) \simeq \mathrm{LMod}(\mathbb{1}[M])$$

The left-hand side consists of objects endowed with an M-action, while the right-hand side consists of modules over the algebra $\mathbb{1}[M]$ in the symmetric monoidal category $\mathcal{C}$.

As a corollary, we have for instance an identification

$$\mathrm{Fun}(X, \mathrm{Sp}) \simeq \mathrm{LMod}(\mathbb{S}[\Omega X])$$

for any connected pointed groupoid X.

Cohomology of groups. If E is an $\mathbb{S}$-algebra and G is a group object in Gpd, then

$$\begin{aligned} \mathrm{C}^{-*}(\mathrm{B}G; E) &\simeq \mathrm{Hom}_{\mathbb{S}}(\Sigma^\infty_+ \mathrm{B}G, E) \\ &\simeq \mathrm{Hom}_{\mathbb{S}}(\mathbb{S} \otimes \mathrm{B}G, E) \\ &\simeq \mathrm{Hom}_E(E \otimes \mathrm{B}G, E) \\ &\simeq \mathrm{Hom}_{\mathrm{Fun}(\mathrm{B}G, \mathrm{LMod}(E))}(\underline{E}, \underline{E}) \\ &\simeq \mathrm{Hom}_{E[G]}(E, E) \end{aligned}$$

and in particular

$$\mathrm{H}^*(\mathrm{B}G; E) \simeq \mathrm{Ext}^*_{E[G]}(E, E)$$

Equivalently

$$\mathrm{H}^*(X; E) \simeq \mathrm{Ext}^*_{E[\Omega X]}(E, E)$$

for any pointed connected groupoid X.

Exercise 1. For $n \geq 1$, observe that

$$\begin{aligned} \mathrm{Th}(\mathrm{T}S^n) &\simeq \mathrm{Th}_*(\Sigma \mathrm{T}S^n) \\ &\simeq \mathrm{Th}_*\big(\varepsilon^{\oplus(n+1)}\big) \end{aligned}$$

and the sections of the trivial bundle $\varepsilon^{\oplus(n+1)} \simeq \mathrm{pr}_2 \colon S^n \times S^n \to S^n$ obtained by stabilizing the tangent bundle fiberwise are the maps $\Delta :\simeq (\mathrm{id}, \mathrm{id})$ and (α, id) where $\alpha \colon S^n \simeq S^n$ is the antipodal map.

In particular the Thom space sits canonically inside a cocartesian square

$$\begin{tikzcd} \mathrm{T}S^n \arrow[r] \arrow[d] & S^n \arrow[d, "\Delta"] \\ S^n \arrow[r, "(\alpha,\mathrm{id})"] \arrow[d] \arrow[ur, phantom, "\ulcorner"] & S^n \times S^n \arrow[d, "p"] \\ * \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathrm{Th}(\mathrm{T}S^n) \end{tikzcd}$$

Since $(\alpha, \mathrm{id}) \colon S^n \to S^n \times S^n$ is right inverse to the second projection, the long exact sequence in integral cohomology splits and yields an exact sequence of graded rings

$$0 \longrightarrow \mathrm{H}^*(\mathrm{T}S^n) \xrightarrow{p^*} \mathrm{H}^*(S^n) \otimes \mathrm{H}^*(S^n) \xrightarrow{(\alpha,\mathrm{id})^*} \mathrm{H}^*(S^n) \longrightarrow 0$$

In particular $\mathrm{H}^n(\mathrm{T}S^n)$ is identified with the kernel of the morphism

$$\mathrm{H}^n(S^n) \oplus \mathrm{H}^n(S^n) \longrightarrow \mathrm{H}^n(S^n)$$

sending $[S^n]$ and $[S^n]$ to $(-1)^{n+1}[S^n]$ and $[S^n]$ respectively. Observe now that the map

$$S^n \simeq \mathrm{Th}(S^{n-1} \to *) \to \mathrm{Th}(\mathrm{T}S^n)$$

is given by the following composite[1]

$$S^n \xrightarrow{(\mathrm{id},*)} S^n \times S^n \xrightarrow{p} \mathrm{Th}(\mathrm{T}S^n)$$

[1] If we had chosen to present $\mathrm{Th}(\mathrm{T}S^n)$ as $\big(S^n \times S^n\big)/\Delta$, the map p would be different and we would have had to use $(\alpha, *) \to S^n \times S^n$ instead.

This gives the sign of the Thom class, namely

$$p^*u(\mathrm{T}S^n) = [S^n] + (-1)^n [S^n]$$

and therefore

$$\begin{aligned} e(\mathrm{T}S^n) &= \Delta^* p^* u(\mathrm{T}S^n) \\ &= (1 + (-1)^n)[S^n] \end{aligned}$$

Exercise 2. We show by induction on $d \geq 1$ that

$$\mathrm{H}^*(\mathrm{BSO}(d); \mathbb{F}_2) \simeq \mathbb{F}_2[w_2, \dots, w_d]$$

which implies $\mathrm{H}^*(\mathrm{BSO}; \mathbb{F}_2) \simeq \mathbb{F}_2[w_2, w_3, \dots]$ by passing to the limit.

- observe that $\mathrm{BSO}(1) \simeq *$ has cohomology ring $\mathbb{F}_2$
- assuming that the statement holds at rank d, consider the (oriented) fiber sequence

  $$S^d \simeq \mathrm{SO}(d+1)/\mathrm{SO}(d) \longrightarrow \mathrm{BSO}(d) \xrightarrow{p} \mathrm{BSO}(d+1)$$

  and the induced Gysin sequence (with $\mathbb{F}_2$-coefficients)

  $$\cdots \longrightarrow \mathrm{H}^{k-d-1}(\mathrm{BSO}(d+1)) \xrightarrow{w_{d+1}} \mathrm{H}^k(\mathrm{BSO}(d+1)) \xrightarrow{p^*} \mathrm{H}^k(\mathrm{BSO}(d)) \longrightarrow \cdots$$

  By the induction hypothesis, the graded ring $\mathrm{H}^*(\mathrm{BSO}(d))$ is polynomial on the classes $w_2, \dots, w_d$. Since the Stiefel–Whitney classes are stable, they are pulled back from $\mathrm{H}^*(\mathrm{BSO})$ and in particular from $\mathrm{H}^*(\mathrm{BSO}(d+1))$. This implies that p^* has a section, and we thus obtain a split short exact sequence

  $$0 \longrightarrow \mathrm{H}^*(\mathrm{BSO}(d+1))[d+1] \xrightarrow{w_{d+1}} \mathrm{H}^*(\mathrm{BSO}(d+1)) \xrightarrow{p^*} \mathrm{H}^*(\mathrm{BSO}(d)) \longrightarrow 0$$

  of graded rings. It follows by induction that the canonical map

  $$\mathbb{F}_2[w_2, \dots, w_{d+1}] \to \mathrm{H}^*(\mathrm{BSO}(d+1))$$

  is an isomorphism.

Exercise 3. We prove by induction on d that there exists classes $x_k \in \mathrm{H}^{4k}(\mathrm{BSp})$ for $k \geq 1$ and canonical identifications

$$\mathrm{H}^*(\mathrm{BSp}(d); \mathbb{Z}) \simeq \mathbb{Z}[x_1, \dots, x_d]$$

of graded rings. Observe that it is sufficient to construct x_d in $\mathrm{H}^{4d}(\mathrm{BSp}(d))$ at each stage $d \geq 1$ since the structural map $\mathrm{BSp}(d) \to \mathrm{BSp}$ is $(4d+3)$-connected[2].

- for the base case $d = 1$, observe that $\mathrm{BSp}(1) \simeq \mathbb{P}^\infty(\mathbb{H})$ has cohomology ring $\mathbb{Z}[t]$ with a generator t in degree 4. Since the integral Euler class x_1 of the oriented spherical fibration

  $$S^3 \longrightarrow * \longrightarrow \mathrm{BSp}(1)$$

  is also a generator of $\mathrm{H}^4(\mathrm{BSp}(1)) \simeq \mathbb{Z}$, it follows that

  $$\mathrm{H}^*(\mathrm{BSp}(1)) \simeq \mathbb{Z}[x_1]$$

- assuming that the statement holds at rank d, consider the following fiber sequence

  $$S^{4d+3} \simeq \mathrm{Sp}(d+1)/\mathrm{Sp}(d) \longrightarrow \mathrm{BSp}(d) \xrightarrow{p} \mathrm{BSp}(d+1)$$

  which is canonically oriented since $\mathrm{BSp}(d+1)$ is simply connected. Define

  $$x_{d+1} \in \mathrm{H}^{4(d+1)}(\mathrm{BSp}(d+1)) \simeq \mathrm{H}^{4(d+1)}(\mathrm{BSp})$$

  to be its integral Euler class, and consider the Gysin sequence

[2] This is exactly where the same argument with BSO instead breaks down, since the map $\mathrm{BSO}(d) \to \mathrm{BSO}$ is only d-connected.

$$\cdots \longrightarrow \mathrm{H}^{k-4(d+1)}(\mathrm{BSp}(d+1)) \xrightarrow{x_{d+1}} \mathrm{H}^k(\mathrm{BSp}(d+1)) \xrightarrow{p^*} \mathrm{H}^k(\mathrm{BSp}(d)) \longrightarrow \cdots$$

By the induction hypothesis, the graded ring $\mathrm{H}^*(\mathrm{BSp}(d))$ is polynomial on the classes $x_1, \dots, x_d$ all living in $\mathrm{H}^*(\mathrm{BSp}(d+1))$. In particular p^* admits a section and we get from the above a split exact sequence

$$0 \longrightarrow \mathrm{H}^*(\mathrm{BSp}(d+1))[4(d+1)] \xrightarrow{x_{d+1}} \mathrm{H}^*(\mathrm{BSp}(d+1)) \xrightarrow{p^*} \mathrm{H}^*(\mathrm{BSp}(d)) \longrightarrow 0$$

of graded rings. It follows by induction on the degree that the map

$$\mathbb{Z}[x_1, \dots, x_{d+1}] \to \mathrm{H}^*(\mathrm{BSp}(d+1))$$

is an isomorphism.

We can now tackle the reminding questions.

1) Passing to the limit, we get

$$\mathrm{H}^*(\mathrm{BSp}) \simeq \mathbb{Z}[x_1, x_2, \dots]$$

Observe that the reduction modulo 2 of x_d for some $d \geq 1$ is the unoriented Euler class of the spherical fibration $\mathrm{BSp}(d-1) \to \mathrm{BSp}(d)$. In other words, the map

$$\mathrm{H}^*(\mathrm{BSp}) \to \mathrm{H}^*(\mathrm{BSp}; \mathbb{F}_2)$$

sends each x_d to w_{4d}, where the Stiefel–Whitney classes are pulled back along the map

$$\mathrm{BSp} \simeq \operatorname*{colim}_d \mathrm{BSp}(d) \to \operatorname*{colim}_d \mathrm{BO}(4d) \simeq \mathrm{BO}$$

2) The forgetful map $\mathfrak{v}\colon \mathrm{BSp} \to \mathrm{BU}$ sits for all $d \geq 1$ inside a commutative square

$$\begin{tikzcd} \mathrm{BSp}(d) \arrow[r, "\mathfrak{v}"] \arrow[d] & \mathrm{BU}(2d) \arrow[d] \\ \mathrm{BSp} \arrow[r, "\mathfrak{v}"] & \mathrm{BU} \end{tikzcd}$$

The map $\mathrm{BSp}(d) \to \mathrm{BSp}$ is $(4d+3)$-connected, so induces an isomorphism on $\mathrm{H}^{4d}(-)$. For the purpose of computing $\mathfrak{v}^*(c_{2d})$, it is thus sufficient to work with $\mathfrak{v}\colon \mathrm{BSp}(d) \to \mathrm{BU}(2d)$. We have a cartesian square[3]

$$\begin{tikzcd} S^{4d-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{BSp}(d-1) \arrow[r] \arrow[d, "p"] \arrow[dr, phantom, "\lrcorner"] & \mathrm{BU}(2d-1) \arrow[d, "q"] \\ * \arrow[r] & \mathrm{BSp}(d) \arrow[r, "\mathfrak{v}"] & \mathrm{BU}(2d) \end{tikzcd}$$

and therefore

$$\begin{aligned} \mathfrak{v}^*(c_{2d}) &= \mathfrak{v}^* e(q) \\ &= e(\mathfrak{v}^* q) \\ &= e(p) \\ &= x_d \end{aligned}$$

Finally

$$\mathfrak{v}^*(c_{2d}) = x_d \quad \text{and} \quad \mathfrak{v}^*(c_{2d+1}) = 0$$

since $\mathrm{H}^{4d+2}(\mathrm{BSp}) \simeq 0$.

3) We now turn our attention to $\mathfrak{h}\colon \mathrm{BU} \to \mathrm{BSp}$. Observe that the composite

$$\mathrm{BU} \xrightarrow{\mathfrak{h}} \mathrm{BSp} \xrightarrow{\mathfrak{v}} \mathrm{BU}$$

[3]This square of connected groupoids commutes, and must therefore be cartesian since fibers over the base point are identified.

classifies the assignment $p \mapsto p \otimes_{\mathbb{C}} \mathbb{H} \simeq p \oplus \overline{p}$. In particular for $d \geq 1$:

$$\mathfrak{h}^*\mathfrak{v}^*(c_d) = \sum_{i=0}^{d} (-1)^i c_i c_{d-i}$$

and this sum cancels when d is odd by the change of variables $i := d - i$. By the computation above, we also obtain the formula

$$\mathfrak{h}^*(x_d) = \sum_{i=0}^{2d} (-1)^i c_i c_{2d-i}$$

4) The forgetful map $\mathfrak{w} \colon \mathrm{BSp} \to \mathrm{BO}$ factors as

$$\mathrm{BSp} \xrightarrow{\mathfrak{v}} \mathrm{BU} \xrightarrow{\mathfrak{u}} \mathrm{BO}$$

and therefore

$$\begin{aligned} \mathfrak{w}^*(p_d) &= \mathfrak{v}^*\mathfrak{u}^*(p_d) \\ &= \mathfrak{v}^*\Big(\sum_{i=0}^{2d} (-1)^{i+d} c_i c_{2d-i}\Big) \\ &= (-1)^d \sum_{i=0}^{d} x_i x_{d-i} \end{aligned}$$

Exercise 4. We begin by proving the result for X n-truncated for some $n \geq 1$ by induction:

- for $n = 1$, then $X \simeq \mathrm{B}G$ for some finite group G. By the above discussion, we know that

$$\mathrm{H}^*(\mathrm{B}G; \mathbb{Q}) \simeq \mathrm{Ext}^*_{\mathbb{Q}[G]}(\mathbb{Q}, \mathbb{Q})$$

  Notice that $\mathbb{Q}$ is $\mathbb{Q}[G]$-projective by considering the following retract diagram

$$\mathbb{Q} \xrightarrow{N} \mathbb{Q}[G] \xrightarrow{\varepsilon} \mathbb{Q}$$

  where N is defined by

$$N(1) := \frac{1}{\operatorname{card} G} \sum_{g \in G} g$$

  It follows that $\mathrm{H}^*(\mathrm{B}G; \mathbb{Q}) \simeq \mathbb{Q}$.

- assume that the results holds for some $n \geq 1$, and suppose X to be $(n+1)$-truncated. Distinguish between two cases:

  (1) if $X \equiv \mathrm{K}(G, n+1)$, then consider the following fiber sequence

$$\mathrm{K}(G, n) \longrightarrow * \longrightarrow \mathrm{K}(G, n+1)$$

  Since the pullback map

$$\mathrm{H}^*(*; \mathbb{Q}) \to \mathrm{H}^*(\mathrm{K}(G, n); \mathbb{Q}) \simeq \mathbb{Q}$$

  is an isomorphism by the induction hypothesis, Leray–Hirsch gives a $\mathbb{Q}$-linear identification

$$\begin{aligned} \mathrm{H}^*(\mathrm{K}(G, n+1); \mathbb{Q}) &\simeq \mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q} \\ &\simeq \mathbb{Q} \end{aligned}$$

  (2) in general X is connected, thus non-empty. Choosing some base point, consider the following fiber sequence

$$\mathrm{K}(G, n+1) \longrightarrow X \longrightarrow \tau_{\leq n} X$$

where $G := \pi_{n+1}(X)$ is a finite abelian group. The induced map in cohomology

$$\mathrm{H}^*(X;\mathbb{Q}) \to \mathrm{H}^*(\mathrm{K}(G, n+1);\mathbb{Q}) \simeq \mathbb{Q}$$

is surjective by case (1). Using Leray–Hirsch and the induction hypothesis, we obtain an isomorphism

$$\mathrm{H}^*(X;\mathbb{Q}) \simeq \mathbb{Q}$$

of $\mathbb{Q}$-modules.

In the general case, the map $X \to \tau_{\leq n} X$ is n-connected for $n \geq 0$. In particular it induces isomorphisms in cohomology groups in some range by Hurewicz theorem, so that

$$\begin{aligned}
\mathrm{H}^*(X;\mathbb{Q}) &\simeq \lim_n \mathrm{H}^*(\tau_{\leq n} X;\mathbb{Q}) \\
&\simeq \lim_n \mathbb{Q} \\
&\simeq \mathbb{Q}
\end{aligned}$$